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On a generally covariant quantum theory*

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SUMMARY

A theory, expressed ab initio in generally covariant form, and satisfying the usual quantum postulates, is outlined. The basis for the theory is the structure of the general coordinate transformation group and a way of representing it introduced by Klein [1]. Various representations of the general coordinate transformation group are discussed. As a particular example the mixed second rank tensor density (weight $\frac{1}{2}$) field is treated in some detail due to its possible connexion with gravitation.

I. Introduction

In the past, many methods have been tried for combining general relativity and quantum theory. In these attempts, one has, naturally, started from classical gravitation theory and applied various quantization procedures. It seems as if this kind of approach towards a general relativistically covariant quantum theory should not be too far from success, in spite of all the difficult problems in connexion with it.

Instead of starting with a definite classical theory, we will try in this paper to construct a covariant quantum theory based on the structure of the group of general coordinate transformations. This will be done using ideas developed by Klein [1].

In section II, some general properties of the coordinate transformation group are summarized. In [1] the group considered differed in some respects from the coordinate transformation group;[1] we shall therefore express, in sections III and IV, parts of [1] in terms of the latter group. In these sections, a particular method for the construction of unitary repesentations of the coordinate transformation group is given.

Sections V and VI, finally, are devoted to a discussion of gravitation-like representations.

II. General properties of the coordinate transformation group

Let us consider the group of general coordinate transformations[2]

$$x' = a(x) = x + \alpha(x), \qquad \left|\frac{\partial x'}{\partial x}\right| \neq 0, \tag{1}$$

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[1] Specifically, the Jacobian need not be unity and the coefficients $\alpha^{\mu}{}_{,\nu}$ of the transformation (see formula (1)) are not all independent here.

[2] Here, and often in what follows, we omit the indices on the coordinates.

and a unitary representation of it composed of matrices $D[\alpha(x)]$, where D is a functional of the functions $\alpha(x)$.

The tangent vectors, $Q_\mu(x)$, at the unit element of the group are defined by

$$Q_\mu(x) \equiv \left. \frac{\delta D}{\delta \alpha^\mu(x)} \right|_{\alpha=0}. \tag{2}$$

The right-hand side of (2) is a functional derivative with respect to the functions $\alpha^\mu(x)$.

The $Q_\mu(x)$ are antihermitian because D is unitary. They transform as components of a covariant vector density (see Appendix 1) and fulfill the following Lie integrability conditions

$$[Q_\mu(x'), Q_\nu(x'')] = -Q_\nu(x')\,\delta_{,\mu}(x'-x'') + Q_\mu(x'')\,\delta_{,\nu}(x''-x'). \tag{3}$$

This formula is derived in Appendix 1. A straightforward calculation shows that it is covariant.[1]

The reason why unitary representations of the group are interesting is well known from discussions of the Lorentz group (see for instance Bargmann & Wigner [2]). To each transformation (1), there must exist an active transformation of the quantum mechanical state vector

$$\Psi' = D[\alpha(x)]\,\Psi. \tag{4}$$

The requirement that the inner product[2] $(\Psi, \Psi) > 0$ be invariant under the transformation shows that D is unitary apart from a similarity transformation.[3] Hence the set of matrices $D[\alpha(x)]$ is a unitary representation of the group.

In the following section, a method will be given for the construction of such representations or, what is the same, sets of antihermitian operators $Q_\mu(x)$ satisfying (3).

III. The basic fields

Following [1] we introduce certain "basic fields" satisfying simple commutation relations and construct from these fields operators Q satisfying our requirements. The procedure is best shown by examples. Let us, as a first example, choose one closely connected with the example treated in [1].

As basic fields, we introduce one covariant vector field operator, $\phi_\mu(x)$, and one contravariant vector density field operator, $\pi^\mu(x)$. They are both assumed to be hermitian and to fulfill the commutation relations

$$[\phi_\mu(x'), \phi_\nu(x'')] = 0, \tag{5}$$

$$[\pi^\mu(x'), \pi^\nu(x'')] = 0, \tag{6}$$

$$[\phi_\mu(x'), \pi^\nu(x'')] = i\,\delta^\nu_\mu\,\delta(x'-x''). \tag{7}$$

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[1] The author wishes to thank Dr. B. DeWitt for clearing up a mistake in this connexion.

[2] We exclude indefinite metric, though this case is a possibility when the physical vector space is a proper subspace of the space considered (see sec. IV).

[3] In the following D is always taken to be unitary.

In [1], π^μ was assumed to be an ordinary contravariant vector and (7) was not covariant under transformations with Jacobians different from unity. Due to the density property of π^μ, (5)–(7) are here covariant under arbitrary coordinate transformations. One can show by straightforward computations that the $Q_\mu(x)$ in the expression[1]

$$Q_\mu = -i(\phi_\mu \pi^\nu)_{,\nu} + i\phi_{\nu,\mu}\pi^\nu \tag{8}$$

are antihermitian and satisfy the commutation relations (3). In fact, by calculating

$$\delta\phi_\mu(x) = \int [\phi_\mu(x), Q_\nu(x')]\,\alpha^\nu(x')\,dx' \tag{9}$$

and

$$\delta\pi^\mu(x) = \int [\pi^\mu(x), Q_\nu(x')]\,\alpha^\nu(x')\,dx', \tag{10}$$

and considering $\alpha^\nu(x')$ as infinitesimal, one finds the active variations of ϕ_μ and π^μ corresponding to infinitesimal coordinate transformations. When put into (9), the first term in (8) gives the contribution from the vector transformation while the second one gives the translation term. The corresponding terms for other kinds of fields satisfying relations like (5)–(7) are easily found. Thus one can find a great number of antihermitian representations of (4). Two more examples will be given later.

By performing the differentiation in the first term of (8), one finds that

$$Q_\mu = -i\phi_\mu \pi^\nu{}_{,\nu} + i(\phi_{\nu,\mu} - \phi_{\mu,\nu})\pi^\nu. \tag{11}$$

This shows explicitly the covariance of the equation.[2]

IV. The physical fields

So far nothing has been explicitly said about the nature and dimensionality of the space in which we perform the coordinate transformation but the idea is of course to apply what we have done to the four-dimensional space-time continuum. One then immediately notices that all kinds of dynamical operator equations of the "basic fields" must contradict the commutation relations (5)–(7) which state that any field quantity at one space-time point commutes with all the field quantities at any other space-time point. Hence the basic fields (e.g. ϕ_μ and π^μ) cannot be physical fields (e.g. vector meson fields). Now, the physical fields are not directly observables in quantum field theories and no damage is done to the theory if they are replaced by other quantities which may be easier to handle; but still the following question remains to be answered: How does dynamics enter the theory? An answer was given in [1] by using subsidiary conditions to pick out from the state vector space a subspace containing the dynamically realizable state vectors (or physical state vectors). We will denote such a vector by Ψ_p.

It is clear that also the Ψ_p vectors must form a representation of the transformation group, i.e. the physical subspace must be an invariant subspace. The simplest way of obtaining this is to use a subsidiary condition of the form

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[1] When the same coordinate argument appears everywhere in an equation, we do not as a rule write it down explicitly. Symmetrisation of terms with noncommuting factors in (8) and in similar expressions in the following is understood.

[2] The expression in [1] corresponding to (8) differs from it due to the differences between the respective groups (see footnote (¹) p. 237).

$$A\Psi_p = 0, \tag{12}$$

where A is a covariant set of quantities constructed from the basic fields.[1]

As to the question of what are here the observable quantities, it can be said that they must have, among other properties, the property that, when they operate on a physical vector, they transform it into another physical vector. For the interpretation of the theory, such operators must be constructed from the basic fields. We already know one example of such quantities. They are the Q_μ operators in (11). With A being a covariant, it is immediately clear that they transform the Ψ_p vectors of (12) into each other, due to their property of generating the coordinate transformations (9) and (10).

V. Second rank tensors as basic fields

In order to come a little closer to gravitation, let us try to choose as basic fields one covariant symmetric tensor $h_{\mu\nu}$ and one contravariant symmetric tensor density $f^{\mu\nu}$. They are both assumed to be hermitian and to satisfy the following covariant commutation relations

$$[h_{\mu\nu}(x'), h_{\varkappa\lambda}(x'')] = 0, \tag{13}$$

$$[f^{\mu\nu}(x'), f^{\varkappa\lambda}(x'')] = 0, \tag{14}$$

$$[h_{\mu\nu}(x'), f^{\varkappa\lambda}(x'')] = i(\delta^\varkappa_\mu \delta^\lambda_\nu + \delta^\varkappa_\nu \delta^\lambda_\mu)\,\delta(x'-x''). \tag{15}$$

Using the general rules for constructing the Q_μ field given in connexion with the vector field example, one obtains in this case

$$Q_\mu = -i h_{\mu\varkappa} f^{\varkappa\lambda}{}_{,\lambda} - \tfrac{1}{2} i \,(h_{\mu\varkappa,\lambda} + h_{\mu\lambda,\varkappa} - h_{\lambda\varkappa,\mu})\, f^{\varkappa\lambda}. \tag{16}$$

Using the $h_{\mu\nu}$ field like the metric to lower indices and to define covariant derivatives, (16) can be written

$$Q_\mu = -i\, f^{\sigma}_{\mu;\sigma}, \tag{17}$$

which expression is explicity covariant. So far everything is easy but the formulation of the subsidiary conditions remains to be done. It is well known from linear quantum field theories that a subsidiary condition like (12) must contain only annihilation operators. Otherwise difficulties arise concerning the normalization of the state vectors.[2] There is no reason to believe that we are better off in a theory of the kind outlined here. Klein used only annihilation operators in his formulation of the subsidiary conditions in the Maxwell and Dirac cases (there he did not require general invariance and the conditions were linear).

It is, however, not possible in any simple fashion to combine our $h_{\mu\nu}$ and

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[1] (12) was used in [1] in connexion with Maxwell and Dirac fields. A was there taken to be the annihilation part of the left side of the field equations.

[2] See e.g. [3] p. 100.

$f^{\mu\nu}$ fields to obtain a field and its hermitian conjugate field, which satisfy the usual commutation relations for annihilation and creation operators.[1] This is so because $h_{\mu\nu}$ and $f^{\mu\nu}$ transform differently.

As a matter of fact this difficulty does show up already in the Maxwell case, but in a milder form. It is responsible for the well-known and troublesome fact that, for the fourth component of the vector potential, annihilation and creation operators are interchanged.

The only simple way out of this seems to be to use as basic field not $h_{\mu\nu}$, but a field C_μ^ν transforming as a mixed tensor density of weight one half. We assume C_μ^ν to be hermitian and consider it as the sum of a non-hermitian field $\frac{1}{\sqrt{2}} B_\mu^\nu$ and its hermitian conjugate field

$$C_\mu^\nu = \frac{1}{\sqrt{2}} (B_\mu^\nu + B^{+\nu}{}_\mu). \tag{18}$$

Further we assume the commutation relations

$$[B_\nu^\mu (x'), \quad B_\lambda^\varkappa (x'')] = 0, \tag{19}$$

$$[B^{+\mu}_{\ \nu} (x'), B^{+\varkappa}_{\ \lambda} (x'')] = 0, \tag{20}$$

$$[B_\nu^\mu (x'), \quad B^{+\varkappa}_{\ \lambda} (x'')] = \delta_\lambda^\mu \delta_\nu^\varkappa \delta (x' - x''), \tag{21}$$

which are covariant if the B_ν^μ are assumed to transform like the C_μ^ν. (19)–(21) have exactly the required appearance of commutation relations for annihilation and creation operators.

The Q_μ for this case are

$$Q_\mu = (B^{+\sigma}_{\ \alpha} B_\mu^\alpha)_{,\sigma} - (B^{+\alpha}_{\ \mu} B_\alpha^\sigma)_{,\sigma} + \tfrac{1}{2} B^{+\alpha}_{\ \beta,\mu} B_\alpha^\beta - \tfrac{1}{2} B^{+\beta}_{\ \alpha} B^\alpha_{\beta,\mu}. \tag{22}$$

VI. Dynamical equations for the mixed tensor-case

The dynamics for the C_μ^ν field must be expressed through subsidiary conditions containing B_μ^ν and $B^{+\nu}_{\ \mu}$ (and no other fields; no metric for instance). We also know that these should have some connexion with Einstein's equations.

It is natural then to look first for a set of classical, covariant equations for C_μ^ν related to Einstein's equations. How such a set may be found is shown in a subsequent paper [4]. These equations consist of terms that are products of the C_μ^ν and their derivatives. Let us write them symbolically

$$E(C) = 0. \tag{23}$$

One way of translating equations of this type to quantum mechanics is the following: First we put

$$\Psi^+ N E(C) \Psi = 0, \tag{24}$$

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[1] Here we refer only to the formal property of these operators (see equations (19)–(21)) because no dynamics has been introduced so far.

where N stands for "normal order", i.e. all creation operators are to the left of all annihilation operators. The normal-ordering is an invariant operation (see Appendix 2). We then notice that the terms in $NE(C)$ always appear in pairs in which one of the terms is the hermitian conjugate of the other. Thus we can split $NE(C)$ into two terms

$$NE(C)=e(B,B^{+})+e^{+}(B,B^{+}), \tag{25}$$

where $e(B,B^{+})$ contains, among other terms, the one consisting of B-factors[1] only. One unambiguous and invariant, though somewhat arbitrary, manner of performing this splitting will be shown in an appendix.

As subsidiary condition we now take[2]

$$e(B,B^{+})\Psi=0, \tag{26}$$

from which equation the equation (24) follows according to (25).

VII. Conclusion

The intention of this paper has been to investigate the possibility of constructing a covariant quantum theory along the lines set out in the introduction; so far no definite obstacle has been encountered.

It is tempting to try to found the dynamics not, as above, on correspondence-like assumptions regarding the subsidiary conditions but on irreducible, unitary representations of the general coordinate transformation group, i.e. a procedure similar to the one used for the Lorentz-transformation group (see e.g. [2]). It is probably true that very few of these representations will correspond to physical reality and some kind of selection principle must still be employed. It is possible, however, that this principle can be simply expressed.

The reason why we have not followed this path here is partly due to mathematical difficulties and partly due to the the intricate problem of the physical interpretation in a theory so far removed from the current ones.

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[1] For definition see Appendix 2.

[2] There is no reason to believe that the subsidiary condition must contain annihilation operators *only*, like in the linear case. What we demand is that the last operator on the right hand side in any term should not be a creation operator.

APPENDIX 1

Transformation properties and commutation relations for the Q-field

We write (1) and its inverse transformation

$$x' = a(x) = x + \alpha(x), \tag{A 1}$$

$$x = a^{-1}(x') = x' + \alpha^{-1}(x'). \tag{A 2}$$

Then

$$D^{-1}[\alpha(x)]\, D[\beta(x)]\, D[\alpha(x)] = D[\gamma(x)] \tag{A 3}$$

is correct if

$$\gamma(x) = \alpha(x) + \beta(x + \alpha(x)) + \alpha^{-1}(x + \alpha(x) + \beta(x + \alpha(x))). \tag{A 4}$$

Differentiating (A 3) with respect to $\beta^{\mu}(x')$ and putting $\beta = 0$, we obtain, using (2), the transformation law for the Q-field

$$D^{-1}[\alpha]\, Q_{\mu}(x')\, D[\alpha] = \int Q_{\alpha}(x) \left.\frac{\delta \gamma^{\alpha}(x)}{\delta \beta^{\mu}(x')}\right|_{\beta=0} dx. \tag{A 5}$$

In the right-hand side, we have used the fact that $\gamma = 0$ if $\beta = 0$. This follows from (A 4). Differentiating (A 5) with respect to $\alpha^{\nu}(x'')$ and putting $\alpha = 0$, we obtain the commutation relations

$$[Q_{\mu}(x'), Q_{\nu}(x'')] = \int Q_{\alpha}(x) \left.\frac{\delta^2 \gamma^{\alpha}(x)}{\delta \alpha^{\nu}(x'')\, \delta \beta^{\mu}(x')}\right|_{\alpha=\beta=0} dx. \tag{A 6}$$

What remains is only to calculate the right-hand sides of (A 5) and (A 6) with the help of (A 4). We obtain

$$\frac{\delta \gamma^{\alpha}(x)}{\delta \beta^{\mu}(x')} = \delta^{\alpha}_{\mu}\, \delta(x + \alpha(x) - x') + {\alpha^{-1}}^{\alpha}{}_{,\varrho}(x + \alpha(x) + \beta(x + \alpha(x)))\, \delta^{\varrho}_{\mu}\, \delta(x + \alpha(x) - x'),$$

$$\left.\frac{\delta \gamma^{\alpha}(x)}{\delta \beta^{\mu}(x')}\right|_{\beta=0} = \delta(x + \alpha(x) - x')\, [\delta^{\alpha}_{\mu} + {\alpha^{-1}}^{\alpha}{}_{,\mu}(x + \alpha(x))] \tag{A 7}$$

and

$$\left.\frac{\delta^2 \gamma^{\alpha}(x)}{\delta \alpha^{\nu}(x'')\, \delta \beta^{\mu}(x')}\right|_{\alpha=\beta=0}$$

$$= \delta^{\alpha}_{\mu}\, \delta_{,\sigma}(x - x')\, \delta^{\sigma}_{\nu}\, \delta(x - x'') + \left.\frac{\delta {\alpha^{-1}}^{\alpha}{}_{,\mu}(x)}{\delta \alpha^{\nu}(x'')}\right|_{\alpha=0} \delta(x - x')$$

$$= \delta^{\alpha}_{\mu}\, \delta_{,\nu}(x - x')\, \delta(x - x'') - \delta^{\alpha}_{\nu}\, \delta_{,\mu}(x - x'')\, \delta(x - x'). \tag{A 8}$$

From (A 7) and (A 5) with the substitution

$$y = x + \alpha(x) \tag{A 9}$$

one obtains

$$\begin{aligned} Q'_\mu(x') &= D^{-1}[\alpha]\,Q_\mu(x')\,D[\alpha] \\ &= \int Q_\alpha(y+\alpha^{-1}(y))\,\delta(y-x')\,[\delta^\alpha_\mu + \alpha^{-1\alpha}{}_{,\mu}(y)]\left|\frac{\partial x}{\partial y}\right| dy \\ &= \left|\frac{\partial x}{\partial y}\right|_{y=x'} Q_\alpha(x'+\alpha^{-1}(x'))\,[\delta^\alpha_\mu + \alpha^{-1\alpha}{}_{,\mu}(x')], \end{aligned}$$

where $|\partial x/\partial y|$ stands for the inverse Jacobian of the transformation (A 9). Hence if we define

$$x = x' + \alpha^{-1}(x'), \qquad x' = x + \alpha(x)$$

(previously x has been used only as an integration variable), we obtain

$$Q'_\mu(x') = \left|\frac{\partial x}{\partial x'}\right| \frac{\partial x^\alpha}{\partial x'^\mu}\, Q_\alpha(x). \tag{A 10}$$

(A 10) shows that the Q_α transform together as a covariant vector density. Finally, we use (A 8) in (A 6), which gives

$$[Q_\mu(x'), Q_\nu(x'')] = Q_\mu(x'')\,\delta_{,\nu}(x''-x') - Q_\nu(x')\,\delta_{,\mu}(x'-x'') \tag{A 11}$$

This is the formula (3).

APPENDIX 2

Splitting of a normal-ordered product

It will be shown here that the normal-ordered product $NE(C)$ in (24) can be split up in a sum like (25) consisting of two hermitian conjugate terms, each of which transforms like the covariant $E(C)$.

First we notice that the normal-ordering does not affect the transformation property of $E(C)$. This follows simply by applying the ordering to both members of the transformation formula for $E(C)$. We are going to use the term "B-factor" to mean a B or a derivative of B appearing in a product of B's, B^+'s and their derivatives. The corresponding definition applies to "B^+-factor". A product of the kind mentioned here gives, in general, several terms when transformed, but each of these terms must contain the same number of B-factors as the original product (and consequently also the same number of B^+-factors as the original product).

It is now obvious that the terms in $NE(C)$ which contain a certain number of B-factors must form a sum which transforms exactly like $E(C)$.

An example of the kind of splitting we want is the following: Let $e(B, B^+)$ in (25) be formed by all the terms in $NE(C)$ containing more B-factors than B^+-factors and one half times all the terms containing equally many B- and B^+-factors. The remaining terms form a sum which is the hermitian conjugate of $e(B, B^+)$. (This special splitting is of course to a large extent arbitrary and there might be reasons for using a different one.)

This accomplishes what we desired because apparently $e(B, B^+)$ transforms like $E(C)$.

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